**Boolean Algebra**

**Introduction**

The most obvious way to simplify Boolean expressions is to manipulate them in the same way as normal algebraic expressions are manipulated. With regards to logic relations in digital forms, a set of rules for symbolic manipulation is needed in order to solve for the unknowns.
A set of rules formulated by the English mathematician *George Boole* describe certain propositions whose outcome would be either *true or false*. With regard to digital logic, these rules are used to describe circuits whose state can be either,*1 (true) or 0 (false)*.

**Definition:**

A Boolean Algebra is a mathematical system where two-valued Boolean algebra is defined on a set of 2 elements B = {0,1} with two binary operations OR (+) and AND (•), a unary operation NOT ('), an equality sign (=) to indicate equivalence of expressions, and parenthesis to indicate the ordering of the operations.

**Laws of Boolean Algebra**

Every law has two expressions, (a) and (b). This is known as *duality*. These are obtained by changing every AND(.) to OR(+), every OR(+) to AND(.) and all 1's to 0's and vice-versa.
It has become conventional to drop the **.** (AND symbol) i.e. A**.**B is written as AB.

**Axioms** - need no proof.

1. Closure Property.

The result of each operation is an element of *B*.

2. Identity Element.

a) 1 for AND because x · 1 = 1 · x = x.

b) 0 for OR because x + 0 = 0 + x = x.

3. Commutative Property. From the symmetry of the tables.

a) x · y = y · x.

b) x + y = y + x.

4. Distributive Property.

a) x · (y + z) = (x · y) + (x · z).

b) x + (y · z) = (x + y) · (x + z).

5. Complement Element. For every *x* Î *B*, there exists a complement element *x*' Î *B* such that:

a) *x* + *x*' = 1 0 + 0' = 0 + 1 = 1 and 1 + 1' = 1 + 0 = 1

b) *x* · *x'* = 0 0 · 0' = 0 · 1 = 0 and 1 · 1' = 1 · 0 = 0

6. Cardinality Bound. There are at least 2 elements *x*, *y* Î *B* such that *x* ¹ *y*. 0 ¹ 1.

**Basic Theorems** - need to be proven.

1. Idempotency.

a) *x* + *x* = *x*.

b) *x*· *x = x.*

Proof of 1a)

*x* + *x* = (*x* + *x*) · 1 by Axiom 2a

= (*x* + *x*) (*x* + *x*') by Axiom 5a

= *x* + *xx*' by Axiom 4b

= *x* + 0 by Axiom 5b

= *x* by Axiom 2b

 Proof of 1b)

 x.x=xx+0 by Axiom 2b

 =xx+xx’ by Axiom 5b

 =x(x+x’) by Axiom 4a

 =x.1 by Axiom 5a

 =x by Axiom 2a

2. a) *x* + 1 = 1.

b) *x* · 0 = 0.

 Proof of 2a)

 X+1= 1 . (x+1) by Axiom 2a

 = (x+x’)(x+1) by Axiom5a

 =x+x’.1 by Axiom 4a

 =x+x’ by Axiom 2a

 =1 by Axiom5a

 Proof of 2b)

 x.0=0 by duality

3. Absorption / Redundence

a) *yx* + *x* = *x*

b) (*y* + *x*)*x* = *x*

Proof of 3a)

*yx* + *x* = *yx* + 1*x* by Axiom 2a

= *x*(*y* + 1) by Axiom 4a

= *x*1 by Theorem 2a

= *x* by Axiom 2a

Proof of 3b)

(*y* + *x*)*x*= *xy* + *xx* by Axiom 4a

= xy + *x* by Theorem 1b

= *x* by Theorem 3a

4. Involution.

a) (*x*')' = *x*

5. Associative.

a) (*x* + *y*) + *z* = *x* + (*y* + *z*)

b) *x* (*y z*) = (*x y*) *z*

**De Morgan's Law**.

a) (*x* + *y*)' = *x*' *y*'

b) (*x y*)' = *x*' + *y*'

The algebraic proofs of the associative law and De Morgan’s law are long so their validity is shown by truth tables

Proof of first De Morgan’s law

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| X | Y | X+y | (x+y)’ | X’ | Y’ | X’y’ |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |

Precedence of operators

1. Parentheses
2. NOT
3. AND
4. OR